Analytic Theory for The Determination of Velocity and Stability of Bubbles in a Hele-Shaw Cell

Part II: Stability

S. Tanveer[1]
Mathematics Department
Virginia Polytechnic Institute & State University
Blacksburg, VA 24061

Abstract

Here, we extend the analysis of part I to determine the linear stability of a bubble in a Hele-Shaw cell analytically. Only the solution branch corresponding to largest possible bubble velocity U for given surface tension is found to be stable, while all the others are unstable, in accordance with earlier numerical results.

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1. Mathematical formulation and regular perturbation for the modes

Now we consider the problem of linear stability of a bubble for disturbances that are symmetric about the channel centerline. For the class of disturbances considered, the flux of fluid at infinity is assumed to be the same as in the steady state. This is a realistic assumption for most experiments where the pressure gradient is held fixed. From incompressibility of fluid flow, the bubble area is time independent. Our formulation here is an extension of earlier method for the analytical determination of the zero surface tension symmetric stability modes as shown in section III of Tanveer & Saffman(1987).

Without any loss of generality we set the channel half width a to unity and the fluid velocity at ∞ , V to unity since this is equivalent to nondimensionalization of all relevant variables using combinations of these two quantities. In the frame of the steady bubble translating with constant velocity U , the pressure boundary condition on the bubble boundary is

$$\phi + Ux = \frac{b^2 T}{12\mu} \frac{1}{R} + \text{ constant} \tag{1}$$

where the variables have the same physical meaning as the first part of the paper. The constant appearing on the right hand side of (1) can now depend on time. The kinematic boundary condition that a point on the bubble boundary moves along with the velocity of the fluid at that point can be expressed as

$$\phi_x F_x + \phi_y F_y + F_t = 0 \quad \text{on} \quad F(x, y, t) = 0 \tag{2}$$

where $F(x, y, t) = 0$ determines the bubble boundary $(x(t), y(t))$ at time t . As for the steady problem (Tanveer, 1986), we consider the mapping function $z(\varsigma, t)$ from the interior of the unit ς semi-circle into the fluid flow region exterior of the bubble on one side of the channel centerline ($z = x + iy$) such that $\varsigma = \pm 1$ are mapped to the rear and front stagnation points on the bubble respectively; $\varsigma = \pm\tilde{\alpha}$ are mapped to $z = \mp\infty$. The semi-circular arc corresponds to the bubble boundary, while the real diameter in the ς -plane corresponds to the wall $y = 1$ and the channel centerline $y = 0$. $\tilde{\alpha}$ depends on time and has to be determined. In the linear stability problem, it is convenient to decompose

$$\tilde{\alpha} = \alpha + \alpha_1 \tag{3}$$

where α is time independent and is specified as a parameter characterizing the size of the steady bubble whose stability is being studied. It is convenient to decompose the mapping function into

$$z = z_0(\varsigma, \tilde{\alpha}) + k_s \tilde{f}(\varsigma, t) \tag{4}$$

where $\tilde{f}$ is some analytic function within the unit circle that has to be determined and z_0 is given by

$$z_0(\varsigma,\tilde{\alpha}) = \frac{1}{\pi}\ell n\left(\frac{\varsigma-\tilde{\alpha}}{\varsigma+\tilde{\alpha}}\right) + \frac{1}{\pi}\left(\frac{2}{U}-1\right)\ell n\left(\frac{1+\tilde{\alpha}\varsigma}{1-\tilde{\alpha}\varsigma}\right) \tag{5}$$

and k_s is a constant given by

$$k_s = \frac{2\alpha}{\pi U}[U(1+\alpha^2)-2\alpha^2] \tag{6}$$

We find

$$\frac{dz}{d\varsigma} = k_s(\tilde{f}' + \eta\tilde{h}) \tag{7}$$

where

$$\eta = \frac{\tilde{\alpha}[U(1+\tilde{\alpha}^2)-2\tilde{\alpha}^2]}{\alpha[U(1+\alpha^2)-2\alpha^2]} \tag{8}$$

and the function $\tilde{h}$ is given by

$$\tilde{h} = \frac{(1-\tilde{p}^2\varsigma^2)}{(\varsigma^2-\tilde{\alpha}^2)(1-\tilde{\alpha}^2\varsigma^2)} \quad \text{where} \quad \tilde{p}^2 = \frac{U(1+\tilde{\alpha}^2)-2}{U(1+\tilde{\alpha}^2)-2\tilde{\alpha}^2} \tag{9}$$

We decompose the complex velocity potential $W \equiv \phi + i\,\psi$ into

$$W = W_0(\varsigma,\tilde{\alpha}) + Uk_s\omega(\varsigma,t) \tag{10}$$

where

$$W_0(\varsigma,\tilde{\alpha}) = -\frac{(U-1)}{\pi}\ell n\frac{(\varsigma-\tilde{\alpha})(1-\tilde{\alpha}\varsigma}{(\varsigma+\tilde{\alpha})(1+\tilde{\alpha}\varsigma)} \tag{11}$$

On substituting (10) and (5) into the dynamic boundary condition (1) on the bubble, we find that it is equivalent to

$$Re(\omega+\tilde{f}) = -\gamma\frac{\{1+Re\,\varsigma\frac{d}{d\varsigma}\ell n(\tilde{f}'+\eta\tilde{h})\}}{|\tilde{f}'+\eta\tilde{h}|} \tag{12}$$

on $\varsigma = e^{i\nu}$ where ν is real and in the interval $[0,\ \pi]$ and

$$\gamma = \frac{b^2T}{12\mu Uk_s^2} \tag{13}$$

as in the first part of the paper. Since for symmetric disturbances, the streamlines coincide both with the cell walls and the channel centerline,

$$Im\,\omega = 0 \tag{14}$$

on the real ς diameter. The geometric condition that real ς axis in the interval $[-1, 1]$ corresponds to the wall and the channel centerline is equivalnet to

(15) $$Im\ \tilde{f} = 0$$

on the real ς axis in [-1, 1]. Since $|\varsigma| = 1$ corresponds to the bubble boundary at all times, the kinematic boundary condition (2) is equivalent to the following equation on the unit ς semi-circular arc (see Tanveer & Saffman (1987) for details)

(17) $$Re[\varsigma \frac{dW}{d\varsigma} - \varsigma^* \left(\frac{dz}{d\varsigma}\right)^* z_t]$$

where $*$ indicates complex conjugate and subscript t denotes time derivative. On substituting for z from (4) and W from (10) and using (11), one finds that (16) is equivalent to the following boundary condition on $\varsigma = e^{i\nu}$ ν in the interval $[0, \pi]$:

(17) $$Re[\varsigma\omega' - \varsigma^*(\tilde{f}' + \eta\tilde{h})^*(\frac{z_0}{k_s} + \tilde{f})_\tau] = 0$$

where in the above prime on quantitites ω and f denotes derivatives with respect to ς and subscript τ indicates derivative with respect to τ , where τ is scaled time given by

(18) $$\tau = \frac{Ut}{k_s}$$

For linear stability analysis, we decompose

(19) $$\tilde{f} = f + F$$

where f satisfies the steady state equations (1) and (2) in the first part of the paper. Assuming F and α_1 small compared to f and α , linearization of (12) and (17) about the steady state gives us the following boundary conditions on $\varsigma = e^{i\nu}$:

(20)
$$Re\,(\omega + F) = -\frac{\gamma}{|f' + h|}\{Re\,\varsigma\frac{d}{d\varsigma}\left(\frac{F' + H_1\alpha_1}{f' + h}\right) - Re\left[1 + \varsigma\frac{d}{d\varsigma}\,ln\,(f' + h)\right]\,Re\left[\frac{F' + H_1\alpha_1}{f' + h}\right]\}$$

(21) $$Re\,\left[\varsigma\omega' - \varsigma^* h^*(\frac{z_{0_\alpha}}{k_s}\alpha_{1_\tau} + F_\tau) - \varsigma^* f'^*(\frac{z_{0_\alpha}}{k_s}\alpha_{1_\tau} + F_\tau)\right] = 0$$

where subscript α denotes derivative with respect to α .

(22) $$h(\varsigma) = \frac{1 - p^2\varsigma^2}{(\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2)}$$

and

(23)
$$H_1 = \frac{(1-p^2\varsigma^2)}{(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)}\left[\frac{1}{\alpha}+\frac{2\alpha U(1-\varsigma^2)-4\alpha}{U(1+\alpha^2)(1-\varsigma^2)-2(\alpha^2-\varsigma^2)}+\frac{2\alpha}{\varsigma^2-\alpha^2}+\frac{2\alpha\varsigma^2}{1-\alpha^2\varsigma^2}\right]$$

For ς real in (-1, 1), in addition to (14) substitution of (19) into (15) and use of equation (1) in Part I implies

(24)
$$Im\, F = 0$$

Equations (21), (22), (14) and (24) are the linear stability equations determining the analytic functions ω and F and the time dependent constant α_1. For purposes of analysis of these equations, it is convenient to define analytic functions $\tilde{\kappa}$, $\tilde{R}$ and $\tilde{T}$ within the unit circle with continuity at the boundary such that on the real diameter

(25)
$$Im\, \tilde{\kappa} = 0$$

(26)
$$Im\, \tilde{R} = 0$$

(27)
$$Im\, \tilde{T} = 0$$

and on $\varsigma = e^{i\nu}$,

(28)
$$Re\, \tilde{\kappa} = -\frac{1}{|f'+h|}\left\{Re\, \varsigma\frac{d}{d\varsigma}\left(\frac{F'+H_1\alpha_1}{f'+h}\right) - Re\left[1+\varsigma\frac{d}{d\varsigma}\, ln\, (f'+h)\right]\, Re\left[\frac{F'+H_1\alpha_1}{f'+h}\right]\right\}$$

(29)
$$Re\, \tilde{R} = Re\left[\varsigma^*(h^*+f'^*)\frac{z_{0_\alpha}}{k_s}\right]$$

(30)
$$Re\, \tilde{T} = Re\left[\varsigma^* f'^* F\right]$$

Each of $\tilde{\kappa}$, $\tilde{R}$, $\tilde{T}$ so defined are analytic functions of ς in $|\varsigma| \le 1$ because of smoothness of the right hand side of (28), (29) and (30) on $\varsigma = e^{i\nu}$ and from the application of Schwarz reflection principle for continuation to the lower half ς semi-circle. Substituting for h from (22) into (20) and using the definitions of $\tilde{\kappa}$, $\tilde{R}$ and $\tilde{T}$ it follows that on $\varsigma = e^{i\nu}$

(31)
$$Re\, [\omega + F - \gamma\tilde{\kappa}] = 0$$

(32)
$$Re\left[\varsigma\omega' - \frac{(\varsigma^2-p^2)}{(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)}F_\tau - \tilde{R}\alpha_{1_\tau} - \tilde{T}_\tau\right] = 0$$

By subtracting off the simples poles of the expression within the square parantheses in (32) and adding on regular terms with the same real part, one finds that (32) is equivalent to

(33) $$Re\ \tilde{S} = 0$$

where

(34)
$$\tilde{S} = -\varsigma\omega' + \frac{\varsigma(\varsigma^2 - p^2)}{(1-\alpha^2\varsigma^2)(\varsigma^2-\alpha^2)} F_r + \tilde{R}\alpha_{1_r} + \tilde{T}_r + \frac{(\alpha^2 - p^2)}{2(1-\alpha^4)}(\varsigma^2 - 1)\left[\frac{F_r(\alpha)}{(\varsigma-\alpha)(1-\alpha\varsigma)} \frac{F_r(-\alpha)}{(\varsigma+\alpha)(1+\alpha\varsigma)}\right]$$

Clearly $Im\ \tilde{S} = 0$ for ς real in [-1, 1] because of (14), (24), (26) and (27). Therefore, (33) implies that $\tilde{S} = 0$ for $|\varsigma| \leq 1$ and so

(35)
$$-\varsigma\omega' + \frac{\varsigma(\varsigma^2 - p^2)}{(1-\alpha^2\varsigma^2)(\varsigma^2-\alpha^2)} F_r + \tilde{R}\alpha_{1_r} + \tilde{T}_r + \frac{(\alpha^2 - p^2)}{2(1-\alpha^4)}(\varsigma^2 - 1)\left[\frac{F_r(\alpha)}{(\varsigma-\alpha)(1-\alpha\varsigma)} \frac{F_r(-\alpha)}{(\varsigma+\alpha)(1+\alpha\varsigma)}\right] = 0$$

Again the function within the parantheses on the left hand side of (31) is an anlaytic fucntion in $|\varsigma| \leq 0$ with vanishing imaginary part on real ς axis in $[-1, 1]$ in view of (14), (24) and (25). Thus for any ς with $|\varsigma| \leq 1$,

(36) $$\omega + F - \gamma\,\tilde{\kappa} = 0$$

Equations (35) and (36) hold for any ς with $|\varsigma| > 1$ as well when one obtains proper analytic continuation of each of κ , $\tilde{R}$ and $\tilde{T}$ defined originally in $|\varsigma| \leq 1$. On substituting for ω from (36) into (35) one obtains

(37)
$$\varsigma F' + \frac{\varsigma(\varsigma^2 - p^2)}{(1-\alpha^2\varsigma^2)(\varsigma^2-\alpha^2)} F_r + \tilde{R}\alpha_{1_r} + \tilde{T}_r + \frac{(\alpha^2 - p^2)}{2(1-\alpha^4)}(\varsigma^2 - 1)\left[\frac{F_r(\alpha)}{(\varsigma-\alpha)(1-\alpha\varsigma)} + \frac{F_r(-\alpha)}{(\varsigma+\alpha)(1+\alpha\varsigma)}\right] = \gamma\varsigma\tilde{\kappa}$$

On evaluation of the above equation at $\varsigma = 0$, we find that

(38) $$\frac{(\alpha^2 - p^2)}{2\alpha(1-\alpha^4)}\left[F_r(\alpha) - F_r(-\alpha)\right] + \tilde{T}_r(0) + \tilde{R}(0)\alpha_{1_r} = 0$$

and this determines α_1 . One can use (38) back into (37) to obtain

(39) $$F' + \frac{(\varsigma^2 - p^2)}{(1-\alpha^2\varsigma^2)(\varsigma^2-\alpha^2)} F_r = R_{3_r} + \gamma\tilde{\kappa}'$$

where

(40) $$R_3 = \frac{F(\alpha)\hat{P}_1}{\varsigma - \alpha} + \frac{F(-\alpha)\hat{P}_2}{\varsigma + \alpha} + \frac{\tilde{T}(0)\tilde{R}}{\tilde{R}(0)\varsigma} - \frac{\tilde{T}}{\varsigma}$$

where

$$\hat{P}_1 = \frac{(\alpha^2 - p^2)}{2(1-\alpha^4)} \left[-\frac{\varsigma^2 - 1}{\varsigma(1-\alpha\varsigma)} + \frac{(\varsigma - \alpha)\tilde{R}(\varsigma)}{\alpha\tilde{R}(0)\varsigma} \right] \tag{41}$$

$$\hat{P}_2 = \frac{(\alpha^2 - p^2)}{2(1-\alpha^4)} \left[-\frac{\varsigma^2 - 1}{\varsigma(1+\alpha\varsigma)} - \frac{(\varsigma + \alpha)\tilde{R}(\varsigma)}{\alpha\tilde{R}(0)\varsigma} \right] \tag{42}$$

We note that $\tilde{R}(0) \neq 0$ is a necessary condition for the above to be valid. This conditions is satisfied when $\gamma = 0$ for in that case $\tilde{R} = \tilde{R}_0$, where from (5),(22) and (29), $\tilde{R}_0$ is determined from the boundary condition

$$Re\ \tilde{R}_0 = \frac{2}{\pi k_s} Re \left\{ \frac{\varsigma(\varsigma^2 - p^2)}{(\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2)} \left[-\frac{\varsigma}{\varsigma^2 - \alpha^2} + \frac{\varsigma(\alpha^2 - p^2)}{(1 - \alpha^2 p^2)(1 - \alpha^2\varsigma^2)} \right] \right\} \tag{43}$$

on $\varsigma = e^{i\nu}$ and on the real diameter the imaginary part of $\tilde{R}_0$ vanishes. It is not difficult to see that

$$\begin{aligned}
\tilde{R}_0(\varsigma) \quad = \quad & \frac{2}{\pi k_s} \left\{ \frac{\varsigma^2(\varsigma^2 - p^2)}{(\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2)} \left[-\frac{1}{(\varsigma^2 - \alpha^2)} + \frac{(\alpha^2 - p^2)}{(1 - \alpha^2\varsigma^2)(1 - \alpha^2 p^2)} \right] \right. \\
& + \frac{(\alpha^2 - p^2)}{4(1-\alpha^4)} \left[\frac{1}{(\varsigma - \alpha)^2} + \frac{1}{(\varsigma + \alpha)^2} - \frac{\varsigma^2}{(1 - \alpha\varsigma)^2} - \frac{\varsigma^2}{(1 + \alpha\varsigma)^2} \right. \\
& \left. + \left(\frac{1}{\alpha} + \frac{2\alpha}{\alpha^2 - p^2} + \frac{2\alpha^3}{1 - \alpha^4} \right) \left(\frac{1}{\varsigma - \alpha} - \frac{\varsigma}{1 - \alpha\varsigma} - \frac{1}{\varsigma + \alpha} + \frac{\varsigma}{1 + \alpha\varsigma} \right) \right] \\
& \left. + \frac{\alpha(\alpha^2 - p^2)^2}{2(1-\alpha^4)^2(1 - \alpha^2 p^2)} \left[-\frac{1}{\varsigma - \alpha} + \frac{\varsigma}{1 - \alpha\varsigma} + \frac{1}{\varsigma + \alpha} - \frac{\varsigma}{1 + \alpha\varsigma} \right] \right\}
\end{aligned} \tag{44}$$

and so

$$\tilde{R}_0(0) = \frac{2}{\pi k_s} \left\{ -\frac{(\alpha^2 - p^2)}{(1 - \alpha^4)} \left(\frac{1}{\alpha^2 - p^2} + \frac{\alpha^2}{1 - \alpha^4} \right) + \frac{(\alpha^2 - p^2)^2}{(1-\alpha^4)^2(1 - \alpha^2 p^2)} \right\} \tag{45}$$

which is clearly non zero. For small γ , f is small and therefore one could argue that $\tilde{R}(0) \neq 0$ for sufficiciently small γ . We will assume that is indeed the case. Now we assume that F has a $e^{\sigma t}$ time dependence then the mode corresponding to σ , we can try a regular perturbation expansion of the form

$$F = F_0(\varsigma, \sigma, U, \alpha) \ + \ \gamma F_1(\varsigma, \sigma, U, \alpha), \ + \ ... \tag{46}$$

where for convenience of regular perturbation expansion, U is being treated as a parameter together with α , though the steady state selection implies that U is a function of γ and α . Tanveer & Saffman (1987) calculated the eigenmode F_0 . Using their procedure, it is straighforward though laborious to find the next term F_1 and one finds no restriction on σ . We assume that this is indeed the case to every order in γ .

2. Analytical continuation and reduction to an ODE

We now proceed to extract the leading order transcendentally small terms by analytic continuation of the boundary conditions to the unphysical region $|\varsigma| > 1$. The procedure is in principle very much like the first part of this paper. This illustrates the power of this formalism, which otherwise may appear rather complicated compared to the Combescot-Dombre analysis. The complication here compared to the first part of this paper is that we have two analytic functions f and ω rather than one. We now proceed with the analytic continuation of the equations. Rather than work on the linearized set of equations just derived, it is convenient to analytically continue the full nonlinear equation and carry out the linearization at a later stage.

We can write equation (12) as

(47)
$$Re\left[1+\frac{\varsigma(\tilde{f}''+\eta\tilde{h}')}{(\tilde{f}'+\eta\tilde{h})}+\frac{\tilde{g}}{\gamma}\right]=0$$

where $\tilde{g}$ is an analytic function in the unit semicircle defined by the boundary conditions

(48)
$$Re\ \tilde{g}=Re(\tilde{w}+\tilde{f})|\tilde{f}'+\eta\tilde{h}| \quad \text{on } \varsigma=\rho\ e^{i\nu}$$

and on the real diameter,

(49)
$$Im\ \tilde{g}=0$$

Using Poisson's integral formula relating a harmonic function and its conjugate to the value of the harmonic function on it's boundary, by using the definition of $\tilde{h}$, we find that (48) implies

(50)
$$\tilde{g}(\varsigma)=\tilde{I}(\tilde{f},\omega,\varsigma)\equiv\frac{1}{4\pi i}\oint_{|\varsigma'|=1}\frac{d\varsigma'}{\varsigma'}\left\{\frac{\varsigma+\varsigma'}{\varsigma'-\varsigma}\right\}\frac{\tilde{\ell}_1^{\frac{1}{2}}(\varsigma')\tilde{\ell}_2^{\frac{1}{2}}(\varsigma')}{(\varsigma'^2-\tilde{\alpha}^2)(1-\tilde{\alpha}^2\varsigma'^2)}\left\{\tilde{f}(\varsigma')+\tilde{f}(\frac{1}{\varsigma'})+\omega(\varsigma')+\omega(\frac{1}{\varsigma'})\right\}$$

where

(51)
$$\tilde{\ell}_1(\varsigma)=\eta\varsigma^2(\varsigma^2-\tilde{p}^2)+(\varsigma^2-\tilde{\alpha}^2)(1-\tilde{\alpha}^2\varsigma^2)\tilde{f}'(\frac{1}{\varsigma})$$

and

(52)
$$\tilde{\ell}_2(\varsigma)=\eta(1-\tilde{p}^2\varsigma^2)+(\varsigma^2-\tilde{\alpha}^2)(1-\tilde{\alpha}^2\varsigma^2)\tilde{f}'(\varsigma)$$

Analytic continuation outside the unit circle gives us

(53)
$$\tilde{g}(\varsigma)=\tilde{I}(\tilde{f},\omega,\varsigma)+\frac{\tilde{\ell}_1^{\frac{1}{2}}(\varsigma)\tilde{\ell}_2^{\frac{1}{2}}(\varsigma)}{(\varsigma^2-\tilde{\alpha}^2)(1-\tilde{\alpha}^2\varsigma^2)}\left\{\tilde{f}(\varsigma)+\tilde{f}(\frac{1}{\varsigma})+\omega(\varsigma)+\omega(\frac{1}{\varsigma})\right\}$$

Now, we notice that the quantity within the paranthesis in (47) is an analytic function of ς with simple poles at $\varsigma = \pm\tilde{\alpha}$ and has vanishing imaginary parts on the real diameter. By subtracting off the simple poles and adding on terms with the same real parts on the unit circle, it is easy to conclude that (47) is equivalent to

$$1 + \frac{\varsigma(\tilde{f}'' + \eta\tilde{h}')}{(\tilde{f}' + \eta\tilde{h})} + \frac{\tilde{g}}{\gamma} + \frac{2\varsigma^2}{\varsigma^2 - \tilde{\alpha}^2} - \frac{2}{1 - \tilde{\alpha}^2\varsigma^2} = 0 \tag{54}$$

Equation (54) is valid everywhere in the complex plane. For $|\varsigma| > 1$, we must use the expression (53) rather than (50) for $\tilde{g}$. Using the decomposition (3) and (19) and assuming that each of α_1, F and ω are small, we expand $\tilde{I}$ to linear order to find

$$\tilde{I}(\tilde{f}, \omega, \varsigma) = I(f, \varsigma) + I_5(F, \omega, f, \varsigma) + \alpha_1 I_6(f, \varsigma) \tag{55}$$

where

$$I(f,\varsigma) = \frac{1}{4\pi i}\oint_{|\varsigma'|=1} \frac{d\varsigma'}{\varsigma'}\left[\frac{\varsigma + \varsigma'}{\varsigma' - \varsigma}\right] \frac{\ell_1^{\frac{1}{2}}(\varsigma)')\ell_2^{\frac{1}{2}}(\varsigma')}{(\varsigma'^2 - \alpha^2)(1 - \alpha^2\varsigma'^2)}(f(\varsigma') + f(\frac{1}{\varsigma'})) \tag{56}$$

where

$$\ell_1(\varsigma) = \varsigma^2(\varsigma^2 - p^2) + (\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2)f'(\frac{1}{\varsigma}) \tag{57}$$

$$\ell_2(\varsigma) = (1 - p^2\varsigma^2) + (\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2)f'(\varsigma) \tag{58}$$

and

$$I_5(F, \omega, f, \varsigma) = \frac{1}{4\pi i}\oint_{|\varsigma'|=1} \frac{d\varsigma'}{\varsigma'}\left[\frac{\varsigma + \varsigma'}{\varsigma' - \varsigma}\right]\{K_3(f, \varsigma') \tag{59}$$

$$\left[F(\varsigma') + \omega(\varsigma') + F(\frac{1}{\varsigma'}) + \omega(\frac{1}{\varsigma'})\right] + K_4(f,\varsigma')F'(\varsigma') + K_5(f,\varsigma')F'(\frac{1}{\varsigma'})\}$$

where

$$K_3(f,\varsigma) = \frac{\ell_1^{\frac{1}{2}}(\varsigma)\ell_2^{\frac{1}{2}}(\varsigma)}{(\varsigma^2 - \alpha^2)(1 - \alpha^2\varsigma^2)} \tag{60}$$

$$K_4(f,\varsigma) = \frac{1}{2}\ell_1^{\frac{1}{2}}\ell_2^{-\frac{1}{2}}[f(\varsigma) + f(\frac{1}{\varsigma})] \tag{61}$$

$$K_5(f,\varsigma) = \frac{1}{2}\ell_1^{-\frac{1}{2}}\ell_2^{\frac{1}{2}}[f(\varsigma) + f(\frac{1}{\varsigma})] \tag{62}$$

and

$$I_6(f,\varsigma) = \frac{1}{4\pi i}\oint \frac{d\varsigma'}{\varsigma'}\left[\frac{\varsigma+\varsigma'}{\varsigma'-\varsigma}\right]\frac{\ell_1^{\frac{1}{2}}(\varsigma')\ell_2^{\frac{1}{2}}(\varsigma')(f(\varsigma')+f(\frac{1}{\varsigma'}))}{(\varsigma'^2-\alpha^2)(1-\alpha^2\varsigma'^2)} \tag{63}$$

$$\left[\frac{2\alpha}{\varsigma'^2-\alpha^2}+\frac{2\alpha\varsigma'^2}{1-\alpha^2\varsigma'^2}+\frac{\hat{L}_1(\varsigma',\alpha)}{2\ell_1(\varsigma')}+\frac{\hat{L}_2(\varsigma',\alpha)}{2\ell_2(\varsigma')}\right]$$

where

$$\begin{aligned}\hat{L}_1(\varsigma,\alpha) &= \left\{\frac{1}{\alpha}+\frac{2\alpha(U-2)}{U(1+\alpha^2)-2\alpha^2}\right\}\varsigma^2(\varsigma^2-p^2)\\ &-\varsigma^2\left\{\frac{2U\alpha}{U(1+\alpha^2)-2\alpha^2}-\frac{2\alpha(U(1+\alpha^2)-2)(U-2)}{(U(1+\alpha^2)-2\alpha^2)^2}\right\}\\ &-2\alpha(1-\alpha^2\varsigma^2)f'(\frac{1}{\varsigma})-2\alpha\varsigma^2(\varsigma^2-\alpha^2)f'(\frac{1}{\varsigma})\end{aligned} \tag{64}$$

$$\begin{aligned}\hat{L}_1(\varsigma,\alpha) &= \left\{\frac{1}{\alpha}+\frac{2\alpha(U-2)}{U(1+\alpha^2)-2\alpha^2}\right\}(1-p^2\varsigma^2)\\ &-\varsigma^2\left\{\frac{2U\alpha}{U(1+\alpha^2)-2\alpha^2}-\frac{2\alpha(U(1+\alpha^2)-2)(U-2)}{(U(1+\alpha^2)-2\alpha^2)^2}\right\}\\ &-2\alpha(1-\alpha^2\varsigma^2)f'(\varsigma)-2\alpha\varsigma^2(\varsigma^2-\alpha^2)f'(\varsigma)\end{aligned} \tag{65}$$

Linearizing (54) and using (53) and subtracting off the steady state, we find

$$F''+(P_1+Q_1)F'+\frac{L_1}{\gamma}(F+\omega) = R_2 \tag{66}$$

where

$$R_2 \equiv \alpha_1 R_1(\varsigma)-\frac{(f'+h)}{\gamma\varsigma}I_5(F,\omega,f,\varsigma)-\frac{L_1}{\gamma}\{F(\frac{1}{\varsigma})+\omega(\frac{1}{\varsigma})\}-\frac{1}{\gamma}\frac{(f'+h)}{\varsigma}K_5(f,\varsigma)F'(\frac{1}{\varsigma}) \tag{67}$$

where

$$P_1 = \frac{-(f''+h')}{(f'+h)} = \frac{1}{\varsigma}+\frac{2\varsigma}{\varsigma^2-\alpha^2}-\frac{2}{\varsigma(1-\alpha^2\varsigma^2)}+\frac{g}{\gamma\varsigma} \tag{68}$$

where g is as defined in the first part of the paper and so from equation (20) of Part I,

$$\frac{g}{\gamma\varsigma} = \frac{I(f,\varsigma)}{\gamma\varsigma}+\frac{\ell_1^{\frac{1}{2}}\ell_2^{\frac{1}{2}}}{\gamma\varsigma(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)}[f(\varsigma)+f(\frac{1}{\varsigma})] \tag{69}$$

and

$$Q_1 = \frac{1}{2\gamma}\frac{(f'+h)}{\varsigma}\ell_1^{\frac{1}{2}}(\varsigma)\ell_2^{-\frac{1}{2}}(\varsigma)\{f(\varsigma)+f(\frac{1}{\varsigma})\} \tag{70}$$

and

$$L_1 = \frac{(f'+h)}{\varsigma}\frac{\ell_1^{\frac{1}{2}}\ell_2^{\frac{1}{2}}}{(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)} \tag{71}$$

and

$$\begin{aligned} R_1 \;=\; & \frac{(f'+h)}{\varsigma}\left[-\varsigma\frac{d}{d\varsigma}\left(\frac{H}{f'+h}\right) - \frac{4\alpha\varsigma^2}{(\varsigma^2-\alpha^2)^2} + \frac{4\alpha\varsigma^2}{(1-\alpha^2\varsigma^2)^2}\right. \\ & -\frac{I_6(f,\varsigma)}{\gamma} - \frac{\ell_1^{\frac{1}{2}}(\varsigma)\ell_2^{\frac{1}{2}}(\varsigma)(f(\varsigma)+f(\frac{1}{\varsigma}))}{\gamma(\varsigma^2-\alpha^2)(1-\alpha^2\varsigma^2)} \\ & \left.\left\{\frac{2\alpha}{\varsigma^2-\alpha^2} + \frac{2\alpha\varsigma^2}{1-\alpha^2\varsigma^2} + \frac{\hat{L}_1(\varsigma,\alpha)}{2\ell_1} + \frac{\hat{L}_2(\varsigma,\alpha)}{2\ell_2}\right\}\right] \end{aligned} \tag{72}$$

The linearized kinematic boundary condition is given by (35). We need the analytic continuation of each of $\tilde{T}$ and $\tilde{R}$ outside the unit circle. For $\tilde{T}$ defined by (27) and (30), it follows from Posson's integral formula that for ς inside the unit circle

$$\tilde{T} = \frac{1}{4\pi i}\oint_{|\varsigma'|=1}\frac{d\varsigma'}{\varsigma'}\left(\frac{\varsigma+\varsigma'}{\varsigma'-\varsigma}\right)\left[\frac{1}{\varsigma'}f'\left(\frac{1}{\varsigma'}\right)F(\varsigma') + \varsigma' f'(\varsigma')F\left(\frac{1}{\varsigma'}\right)\right] \equiv I_3(f,F,\varsigma) \tag{74}$$

For $|\varsigma| > 1$, by contour deformation in the ς' plane, we obtain the analytic continuation of (74) to be

$$\tilde{T} = I_3(f,F,\varsigma) + \left[\frac{1}{\varsigma}f'(\frac{1}{\varsigma})F(\varsigma) + \varsigma f'(\varsigma)F(\frac{1}{\varsigma})\right] \tag{75}$$

We noted before that $\tilde{R} = \tilde{R}_0$ given by (44) when $\gamma = 0$, i.e. $f = 0$. So from (26) and (29) and Poisson integral formula it follows that for $|\varsigma| \leq 1$,

$$\tilde{R} = \tilde{R}_0 + I_4(f,F,\varsigma) \tag{76}$$

where

$$I_4(f,F,\varsigma) = \frac{1}{4\pi i k_s}\oint_{|\varsigma'|=1}\frac{d\varsigma'}{\varsigma'}\left(\frac{\varsigma'+\varsigma}{\varsigma'-\varsigma}\right)\left[\frac{1}{\varsigma'}f'(1/\varsigma')z_{0_\alpha}(\varsigma') + \varsigma' f'(\varsigma')z_{0_\alpha}(1/\varsigma')\right] \tag{77}$$

Analytical continuation for $|\varsigma| > 1$ gives us

$$\tilde{R} = \tilde{R}_0 + I_4(f,F,\varsigma) + \frac{1}{k_s}\left(\frac{1}{\varsigma}f'(1/\varsigma)z_{0_\alpha}(\varsigma) + \varsigma f'(\varsigma)z_{0_\alpha}(1/\varsigma)\right) \tag{78}$$

Since equations (35) and (66) do not involve time τ explicitly, we now assume a modal form with a $e^{\sigma\tau}$ time dependence. We note that in (66) the right hand side R_2 involves values of F and ω either on the boundary of the unit circle or inside where correction to the regular perturbation expansion

$$\omega_0(\varsigma,\sigma,U,\alpha)+\gamma\omega_1(\varsigma,\sigma,U,\alpha)+\cdots \tag{79}$$

for ω and the corresponding series for F (given by (46)) for F are transcendentally small. Thus one can replace F and ω on the right hand side of (66) by its regular perturbation expansion. In our case, as far as obtaining leading order transcendental correction, it is only necessary to retain the zeroth order approximation $F = F_0$, $\omega=\omega_0$ in evaluating R_2 . Similarly we can substitute the regular perturbation expansion for F and ω in the integral term I_3 and $I_4(f,F,\varsigma)$ as well as for $F(1/\varsigma)$ and $\tilde{R}$ given in (75) through (78). On carrying out this substitutions, (35) and (66) becomes a set of two ordinary differential equations for unknown modes F and ω : Strictly speaking, we should use different notation for the modes since F and ω were originally used to denote the general time dependent deviation from the steady state. However, from this point onward, we will only be talking about the modes and so in the interest of non- proliferation of symbols, we retain the same symbols for the modes. ¿From the two differential equations thus obtained from (35) and (66), we eliminate the variable F to obtain a single third order equation for ω .

$$\omega'''+\omega''\left(P_1+Q_1+\frac{2\,S'}{S}\right)+\left(\frac{S''}{S}+\frac{S'}{S}(P_1+Q_1)+\frac{L_1}{\gamma}\right)\omega'+\frac{\sigma L_1}{\gamma S}\,\omega=R_4 \tag{80}$$

where

$$S\;=\;\frac{(1-\alpha^2\varsigma^2)(\varsigma^2-\alpha^2)}{(\varsigma^2-p^2)} \tag{81}$$

and

$$R_4=\frac{R_{2\tau}}{S}-R''_{3\tau}-(\frac{2S'}{S}+P_1+Q_1)R'_{3\tau}-\left(\frac{S''}{S}+\frac{S'}{S}(P_1+Q_1)+\frac{L_1}{\gamma}\right)R_{3\tau} \tag{82}$$

where R_2 , R_3 , P_1 , Q_1 and L_1 are given by (67), (40), (68), (70) and (71) respectively.

3. Asymptotic solutions for small γ

For small γ, dominant balance involving terms with factor of $\frac{1}{\gamma}$ must give $\omega = \omega_0$. Carrying this out this out to higher and higher precision only generates the perturbation expansion (79). It is clear that transcendental terms in surface tension must be generated by looking at the homogeneous part of (80) and trying a WKB approach. We find that the leading order transcendental correction away from the immediate neighborhood of the turning points $\varsigma = \pm\frac{1}{p}$ must be linear combinations of

$$\frac{1}{S} L_1^{-3/4} e^{-\frac{1}{2}\int^{\varsigma} d\varsigma' \left(P_1 + Q_1 - \frac{\sigma(\varsigma'^2 - p^2)}{(1-\alpha^2\varsigma'^2)(\varsigma'^2-\alpha^2)}\right)} e^{\pm i\gamma^{-1/2}\int_{1/p}^{\varsigma} L_1^{1/2} d\varsigma} \tag{83}$$

The above asymptotic expressions for two linearly independent solutions to the homogeneous part of (80) simplify considerably, once we realize that the steady state function $f \sim \gamma f_1$ away from the immediate neighborhood of the critical points as pointed out in Part 1. Using the expressions for f_1, one finds that to the same order of approximation to which (83) is valid, the leading order transcendental correction are multiples of ω_{H_1} and ω_{H_2} given by

$$\omega_{H_{1,2}} = \frac{(\varsigma^2 - p^2)^{3/8}}{(1 - p^2\varsigma^2)^{3/8}} \left(\frac{1 + \alpha\varsigma}{1 - \alpha\varsigma}\right)^{d_1} \left(\frac{\varsigma - \alpha}{\varsigma + \alpha}\right)^{d_2} e^{\pm\frac{i}{\gamma^{1/2}}\int_{1/p}^{\varsigma} L^{1/2} d\varsigma'} \tag{84}$$

where

$$d_1 = \frac{\sigma(1 - \alpha^2 p^2)}{4\alpha(1 - \alpha^4)} \tag{85}$$

$$d_2 = \frac{\sigma(\alpha^2 - p^2)}{4\alpha(1 - \alpha^4)} \tag{86}$$

and

$$L(\varsigma) = \frac{(1 - p^2\varsigma^2)^{3/2}(\varsigma^2 - p^2)^{1/2}}{(\varsigma^2 - \alpha^2)^2(1 - \alpha^2\varsigma^2)^2} \tag{87}$$

as before in the case of the steady state. We notice the similarity of (84) with the expressions for g_1 and g_2 in part I (Equations 33,34). Each of ω_{H_1} and ω_{H_2} is transcendentally small or large depending on the sign of $Re\ P$ where, as in part I,

$$P(\varsigma) \equiv i\int_{1/p}^{\varsigma} d\varsigma\ L^{1/2}(\varsigma)$$

Thus the Stokes lines picture of part I of this paper is critically relevant in determining the nature of transcendental correction. Our work in the 1st part of this paper has shown that steady solutions exist only when $\alpha^2 - p^2 > 0$. Further, we found that at sufficiently small surface tension, U on any branch of solution approaches 2 from below and so in the limiting case of small γ, one can assume that p^2 is positive for fixed α. Thus only Fig. 1 in part I of the paper is relevant in the stability analysis in the limit of zero surface tension.

On inclusion of the leading order transcendental correction, the leading order asymptotic behavior of ω in sector I (see Fig. 1) is given by

$$\omega \sim \omega_0 + C_1\omega_{H_1} + C_2\omega_{H_2} \tag{89}$$

As mentioned in section 4 of Part I, $Re\ P$ has a maximum value of $\gamma^{1/2}\beta$ in sector I. Thus in order that each of the multiples of ω_{H_1} and ω_{H_2} be transcendentally small in all of sector I, it is necessary to have

$$C_1\, e^{\beta} = O(1) \tag{90}$$

$$C_2 = O(1) \tag{91}$$

In sector II

$$\omega \sim \omega_0 + C_3\omega_{H_1} + C_4\omega_{H_2} \tag{92}$$

In order that the transcendental corrections in all of sector II be small it is necessary that

$$C_4\, e^{-\gamma^{-1/2}P_m} = O(1) \tag{93}$$

$$C_3 = O(1) \tag{94}$$

In sector III, the leading order behavior is given by

$$\omega \sim \omega_0 + C_5\omega_{H_1} \tag{95}$$

Note that it is not possible to have a multiple of ω_{H_1} in each of (95) since it grows without bounds at $\varsigma = -\alpha$ inside the unit ς semi-circle. In order that (95) be a transcendentally small correction everywhere in sector III of Fig.1, it is necessary that

$$C_5 e^{\gamma^{1/2}P_m} = O(1) \tag{96}$$

Now, we assume that $\alpha - p$ small with α of order unity, as assumed in Section 4b of part I of this paper. As before in part I, section 4b, we now introduce the local variables near $\varsigma = -1/p$ by defining

$$1 + \alpha\varsigma = \epsilon\, x_1 \tag{97}$$

(98) $$f(\varsigma) = \frac{\epsilon\alpha}{1-\alpha^4}\,\tilde{D}(x_1)$$

Then to the leading order in ϵ , equation (80) transforms into

(99) $$\frac{d^3\omega}{dx_1^3} + \left(\frac{3}{x_1} - \frac{3}{2}\beta'\tilde{M}\frac{\tilde{D}}{x_1}\right)\frac{d^2\omega}{dx_1^2} + \left(1 - \frac{3}{2}\beta'\tilde{M}\tilde{D} - \beta'\tilde{M}^3\right)\frac{1}{x_1^2}\frac{d\omega}{dx_1} - \frac{\sigma}{2\alpha}\,\frac{\beta'\tilde{M}^3}{x_1^3}\omega = 0$$

where

(100) $$\tilde{M} = \left(1 + x_1 + x_1\tilde{D}'\right)^{1/2}$$

where

(101) $$\beta' = \frac{\epsilon^{3/2}2^{-1/2}\alpha}{\gamma\left(1-\alpha^4\right)^{3/4}}$$

and the function $\tilde{D}(x_1)$ is determined as described in section 4b of part I. For large x_1 with corresponding ς in sector II of Fig. 1,

(102)

$$\omega(x_1) \sim x_1^{\sigma/(4\alpha)-\frac{3}{8}} + A_1(\beta')x_1^{-3/8+\sigma/(4\alpha)}\,e^{-4\beta'^{1/2}(1+x_1)^{3/4}/3} + A_2(\beta')x_1^{-3/8+\sigma/(4\alpha)}\,e^{4\beta'^{1/2}(1+x_1)^{3/4}/3}$$

and this matches with the expression for ω in (92) in sector II provided

(103) $$\frac{A_1e^{\beta'^{1/2}R}}{C_3}e^{-\frac{P_m}{\gamma^{1/2}}} = \frac{A_2e^{-\beta'^{1/2}R}}{C_4}e^{\frac{P_m}{\gamma^{1/2}}} = e^{i3\pi/4}\alpha^{-3/4}\left(1-\alpha^4\right)^{3/8}\epsilon^{\sigma/(4\alpha)-3/8}2^{-3/8}2^{-\sigma/4\alpha}$$

where R is defined as in section 4b, part I (Equation 83).

For large x_1 with corresponding ς in sector III,

(104)

$$\omega(x_1) \sim x_1^{\sigma/(4\alpha)-\frac{3}{8}} + A_3(\beta')x_1^{-3/8+\sigma/(4\alpha)}\,e^{-4\beta'^{1/2}(1+x_1)^{3/4}/3} + A_4(\beta')x_1^{-3/8+\sigma/(4\alpha)}\,e^{4\beta'^{1/2}(1+x_1)^{3/4}/3}$$

and this matches with (95) provided

(105) $$A_4 = 0$$

and

(106) $$\frac{A_3e^{\beta'^{1/2}R}}{C_3}e^{-\frac{P_m}{\gamma^{1/2}}} = e^{i3\pi/4}\alpha^{-3/4}\left(1-\alpha^4\right)^{3/8}\epsilon^{\sigma/(4\alpha)-3/8}2^{-3/8}2^{-\sigma/4\alpha}$$

We now move to the immediate neighborhood of $\varsigma = \frac{1}{\alpha}$ Introduce local variables

(107) $$(1-\alpha\varsigma) = \epsilon x_2$$

(108) $$f(\varsigma) = \frac{\alpha\epsilon}{(1-\alpha^4)}D_2(x_2)$$

where $D_2(x_2)$ is determined as described in section 4b of Part I. To the leading order in ϵ , equation (80) becomes

$$(109)\quad \frac{d^3\omega}{dx_2^3} + \left(\frac{3}{x_2} - \frac{3}{2}\tilde{\beta}M_2\frac{D_2}{x_2}\right)\frac{d^2\omega}{dx_2^2} + \left(1 - \frac{3}{2}\tilde{\beta}M_2D_2 + \tilde{\beta}{M_2}^3\right)\frac{1}{x_2^2}\frac{d\omega}{dx_2} - \frac{\sigma}{2\alpha}\frac{\tilde{\beta}M_2^3}{x_2^3}\omega = 0$$

For large values x_2 , with corresponding ς in sector II,

(110)

$$\omega(x_2) \sim x_2^{\sigma/(2\alpha)-\frac{3}{8}} + A_5(\tilde{\beta})x_2^{-3/8-\sigma/(4\alpha)}\, e^{-i4\tilde{\beta}^{1/2}(1+x_2)^{3/4}/3} + A_6(\tilde{\beta})x_2^{-3/8+\sigma/(4\alpha)}\, e^{4\tilde{\beta}^{1/2}(1+x_2)^{3/4}/3}$$

and this matches to behavior of solution given in (92) in sector II provided

$$(111)\qquad \frac{A_5e^{i\tilde{\beta}^{1/2}R}}{C_3} = \frac{A_6e^{-i\tilde{\beta}^{1/2}R}}{C_4} = \alpha^{-3/4}(1-\alpha^4)^{3/8}\epsilon^{-\sigma/(4\alpha)-3/8}2^{-3/8}2^{\sigma/4\alpha}$$

For large values of x_2 with corresponding ς in sector I,

(112)

$$\omega(x_2) \sim x_2^{\sigma/(2\alpha)-\frac{3}{8}} + A_7(\tilde{\beta})x_2^{-3/8-\sigma/(4\alpha)}\, e^{-i4\tilde{\beta}^{1/2}(1+x_2)^{3/4}/3} + A_8(\tilde{\beta})x_2^{-3/8+\sigma/(4\alpha)}\, e^{4\tilde{\beta}^{1/2}(1+x_2)^{3/4}/3}$$

and this matches with (89) provided provided

$$(113)\qquad \frac{A_7e^{i\tilde{\beta}^{1/2}J}}{C_1} = \frac{A_8e^{-i\tilde{\beta}^{1/2}J}}{C_2} = \alpha^{-3/4}(1-\alpha^4)^{3/8}\epsilon^{-\sigma/(4\alpha)-3/8}2^{-3/8}2^{\sigma/4\alpha}$$

¿From (94) and (103), it follows that A_1 is transcendentally small compared to A_2 since $e^{\gamma^{-1/2}P_m}$ is transcendentally small and much smaller that terms involvint β' since under the assumption made here, $\alpha - p << 1$. Thus we can neglect A_1 altogether. Then together with (105), this implies that we are interested in solution to (99) so that for large x_1 with corresponding ς in sectors II and III in Fig. 1, the transcendental correction is small. This would determine a unique solution to (99), as will be shortly argued. As far as (109), we note from (93) and (111) that A_6 is transcendentally small compared to A_5 and will therefore be set to 0. This means that we should require that the solution to (109) contain no transcendentally large term for large x_2 when the corresponding ς is in sector II.

We now have to use the condition of smooth bubble back. When σ is real, this would imply that the mode $\omega(\varsigma, \sigma)$ be real in some open interval containing $\varsigma = 1$. From (80), it is clear that if ω is real on such an open interval, then it must be real for ς in $(\alpha, 1/\alpha)$ as all the coefficients of the differential equation are real in that real ς interval. This implies that the solution $\omega(x_2, \sigma)$ of (121) must be real on the postive real x_2 axis. However, when σ is complex, we cannot demand that each mode $\omega(x_2, \sigma)$ should be real on the real positive x_2 axis. The condition of a smooth bubble back is equivalent to requiring that on the positive real x_2 axis

$$\omega(x_2, \sigma) = [\omega(x_2, \sigma^*)]^* \tag{114}$$

On formally taking the complex conjugate of equation (109) on the positive x_2 axis and relating the corresponding solution to the condition (114), it is easily argued that the we must have the property

$$\omega(x_2, \sigma) = [\omega(x_2^*, \sigma^*)]^* \tag{115}$$

for any x_2 not necessarily on the positive real axis. Thus the condition that for large x_2 corresponding to ς in sector II, the solution to (109) does not contain any transcendentally large corrections implies from (115) that for large x_2 with corresponding ς in sector VII (which is a reflection of sector II in Fig. 1) also contain no trancendental large corrections. Indeed, this condition is equivalent to (115) since if we were to require that the asymptotic behavior for large x_2 in sectors II and VII contain no transcendentally large terms, we will be assured that such solution will automatically satisfy the condition (114) and therefore (115) in view of the symmetries of (109). ' But the condition of no transcendentally large terms in two sectors will uniquely determine a solution to (109) as argued earlier for the steady problem in part 1. Note that in the special case when σ is real, such a requirement is indeed equivalent to requiring that $\omega(x_2)$ be real on the real and positive x_2 axis. The actual solution in (109) is not important as it is not needed for the leading order determination of the eigenvalue and the corresponding eigenmodes as the tip and the back problems decouple to the leading order.

To ascertain that the tip of the bubble remains smooth under time dependent symmetric perturbation, we must require that $\omega(\varsigma, \tau)$ as defined originally should be real in some open interval on the real ς axis containing -1. As before with the neighborhood of $\varsigma = +1$, it is easily argued from (80) that if indeed such an open interval exists then $\omega(\varsigma, \tau)$ is analytic in the entire interval $(-1/\alpha, -\alpha)$. However, the same is only true for each of the modes $\omega(\varsigma, \sigma)$ if σ is real. Generally, for complex σ, the condition of smooth tip is equivalent to

$$\omega(x_1, \sigma) = [\omega(x_1, \sigma^*)]^* \tag{116}$$

on the positive real x_1 axis, where $\omega(x_1, \sigma)$ is the solution to (99) satisfying conditions in sectors I and II as mentioned earlier. It is easily argued that the condition (116) is equivalent to requiring that there exists solution to (99) with no transcendentally large term for large x_1 with corresponding ς in sectors I, II, V and VI, where sectors V and VI are the reflections of sectors I and II on the part of the real ς axis between α and $1/\alpha$ (see Fig. 1). From the figure, it is clear that this means that the solution to (99) satisfying the smooth tip condition is equivalent to requiring that there be contain no transcendentally large term for large x_1 in the entire interval $(-\pi, \pi)$. Note that such a requirement is also valid when σ is purely real as can be seen by taking the complex conjugate of equation (99).

We now relate the solution to (99) satisfying the above conditions to a problem which we already solved in the context of the linear stability of a finger (Tanveer, 1987c). We introduce the transformations

$$\xi = -ix_1^{1/2} \tag{117}$$

$$\tilde{D} = 2D \tag{118}$$

$$\beta' = \tilde{\beta}'/4 \tag{119}$$

Then equation (99) becomes

$$\frac{d^3\omega}{d\xi^3} + \frac{1}{\xi}(3 - \frac{3}{2}i\tilde{\beta}'MD)\frac{d^2\omega}{d\xi^2} + \left(1 - \frac{3}{2}i\beta MD + i\tilde{\beta}'M^3\right)\frac{1}{\xi^2}\frac{d\omega}{dxi} + i\frac{\sigma}{\alpha}\frac{\tilde{\beta}'M^3}{\xi^3}\omega = 0 \tag{120}$$

where

$$M = (\xi^2 - 1 - \xi\frac{dD}{d\xi})$$

The requirement of no transcendentally small terms for argument of x_1 in the interval $[-\pi, \pi]$ is equivalent to requiring that in the ξ plane for large argument, there be no transcendentally small terms for argument of ξ in the interval $[-\pi, 0]$. This is precisely the problem that was encountered for the finger (Tanveer, 1987c) and numerical solutions were found for the cases when $\tilde{\beta}'$ was of order unity. Only one branch of solutions for which $\tilde{\beta}'$ was the smallest, i.e. U the largest was found to be stable and all the other branches unstable to tip breaking disturbances. Given the correspondence we have just established, the same must be true for the bubble. For large $\tilde{\beta}'$, (δ in the notation of the Tanveer paper), asymptotic expressions were found for σ and it was found that there were many unstable eigenvalues.

In addition to what has been reported in the Tanveer (1987c) about the limiting values of σ as $\gamma \rightarrow 0$, we note that in the context of the finger eigen values σ for which $Re\, \sigma < 0$ is not physically acceptable since they correspond to singularities in the tail of the finger. In the case of the bubble, these modes physically correspond to disturbances affecting the sides and the back of the bubble, as seen before numerically (Tanveer & Saffman, 1987b). Using the procedure described before (Tanveer, 1987c), we found two nonzero pairs of discrete eigenvalues $\sigma = -1.65 \pm 0.004i$ and $\sigma = -4.1060 \pm 0.11294i$ for the Mclean-Saffman branch.

4. Discussions and Conclusions

Here in this part of the paper, we have demonstrated how the calculation of transcendentally small terms in surface tension is crucial in the proper prediction of eigen values and corresponding eigenmodes for the linear stability of a bubble. We have shown that only one branch of bubble solutions is stable, while all the others are unstable, a result supported by earlier analytical and numerical work on the finger. The next step will be to calculate modal interactions in for calculation of non-linear interactions. It is to be pointed out that in this problem, the calculation of stability and the modal interactions in an analytical procedure is very important since all the usual numerical procedures break down because of inherent ill posedness of the time evolution problem as surface tension tends to zero.

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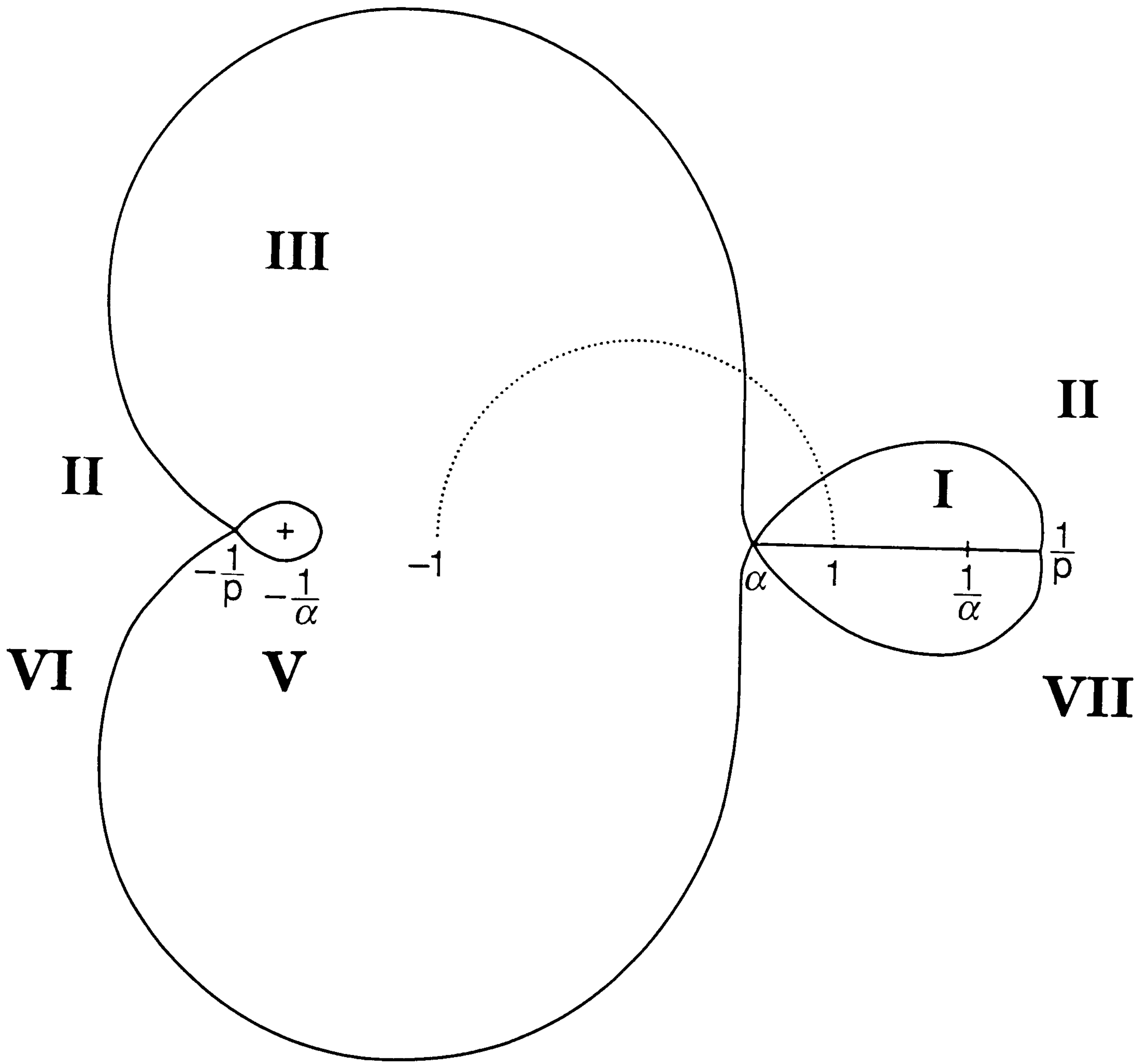

Figure 1 Stokes Lines in the ς plane for $\alpha^2 > p^2 > 0$.

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Theoretical and Computational Fluid Dynamics

Final Report

16. Abstract

Here, we extend the analysis of part I to determine the linear stability of a bubble in a Hele-Shaw cell analytically. Only the solution branch corresponding to largest possible bubble velocity U for given surface tension is found to be stable, while all the others are unstable, in accordance with earlier numerical results.

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two phase flow, pattern formulation, asymptotic theory, stability	67 - Theoretical Mathematics 70 - General Physics Unclassified - Unlimited

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